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Tchebychev surfaces of $\mathbb{S}^3(1)$ with constant curvature functions

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Abstract. We prove that Tchebychev surfaces of $\mathbb{S}^3(1)$ with constant mean curvature or with shape operator of constant squared length are isoparametric. We summarize our results in a survey of types of isoparametric and non-isoparametric Tchebychev surfaces.

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1 Introduction

In the Affine Differential Geometry of hypersurfaces, so called relative Tchebychev hypersurfaces are studied in [5], [6], [7], [8]. In [9], we analogously introduced a new class of hypersurfaces in space forms, again called *Tchebychev hypersurfaces*, and studied this class, especially such hypersurfaces in an Euclidean sphere $\mathbb{S}^{n+1}(1)$. In this paper, we continue this study, in particular that of *Tchebychev surfaces* of $\mathbb{S}^3(1)$, which generalize the class of isoparametric surfaces. We consider, for hypersurfaces with Weingarten operator of maximal *rank*, the first, second and third fundamental forms I, II, III, resp., and their Levi-Civita connections ∇^1, ∇^2, ∇^3, resp. As in the affine theory, we next consider the difference tensor $C := \frac{1}{2}(\nabla^1 - \nabla^3)$ and its traceless part $\widetilde{C}$. In section 3, we use $\widetilde{C}$ to characterize Tchebychev surfaces. From this it follows that surfaces satisfying $C = 0$ or $\widetilde{C} = 0$ give simple examples of Tchebychev surfaces. Next we consider Tchebychev surfaces satisfying special curvature conditions. Our main result with an extrinsic curvature condition is:

Theorem 1.1. *Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be a non-degenerate immersion of a connected, orientable 2-dimensional C^∞-manifold M^2 into $\mathbb{S}^3(1)$. If the immersion x is of Tchebychev type, then the following properties are equivalent:*

(i) *the immersion x has constant mean curvature;*

(ii) *the shape operator associated to x has constant length;*

(iii) *the difference tensor field C vanishes on M^2;*

(iv) *the second fundamental form is parallel with respect to the Levi-Civita connection of the first fundamental form;*

(v) *the immersion x is isoparametric.*

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For the study of an intrinsic curvature condition, we recall a result from [4]: Consider a hypersurface in $\mathbb{S}^{n+1}$ with type number (= *rank* of the Weingarten operator) different from 1 and 2 everywhere. Then we have the following equivalence:

(a) the hypersurface is locally symmetric;

(b) the hypersurface is isoparametric with at most two distinct principal curvatures.

Obviously the condition on the type number excludes the case of surfaces. In [9] we studied the condition (b) above for non-degenerate hypersurfaces of dimension $n \geq 2$ and proved that condition (b) is equivalent to the equation $C = 0$; following [4] for $n \geq 3$, the equation $C = 0$ is then equivalent to the local symmetry. About local symmetry in dimension 2 we have:

Theorem 1.2. *Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be a non-degenerate immersion of M^2 into $\mathbb{S}^3(1)$. Then the following assertions are equivalent:*

(i) *the immersion x is locally symmetric;*

(ii) *the Tchebychev vector field T vanishes on M^2;*

(iii) *the immersion x has constant Gauß-Kronecker curvature.*

We summarize our results in a survey of types of Tchebychev surfaces in section 5 in terms of their principal curvatures and the invariants C, T and $\widetilde{C}$.
For most of the basic notions and facts, we refer to [2], [3], [11], [12]. We use mathematica to simplify calculations.

2 Preliminaries

In this section we summarize basic notions and facts from affine differential geometry of hypersurfaces; we use the notations from [12]. Let M^2 be a connected, orientable C^∞-manifold of dimension $dim M^2 = 2$, $x\colon M^2 \longrightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4$ be an immersion of M^2 into $\mathbb{S}^3(1)$, and y a unit vector field on $\mathbb{S}^3(1)$ normal to M^2. Denote by $\bar{\nabla}$ the Levi-Civita connection of $\mathbb{R}^4$ and by $<,>$ the canonical inner product of the Euclidean structure. To the immersion x are associated three fundamental forms I the first fundamental form (induced metric), II the second and III the third fundamental form related through

$$\mathrm{II}(u,v) = \mathrm{I}(Su, v), \qquad \mathrm{III}(u,v) = \mathrm{I}(Su, Sv), \tag{2.1}$$

where S is the Weingarten (shape) operator and $u, v \in \mathfrak{X}(M^2)$ ($\equiv$ the $C^\infty-$module of vector fields on M^2). The immersion x is said to be non-degenerate or regular if the shape operator S has maximal rank. In this case, the second fundamental form II is a semi-Riemannian metric while the third fundamental form III is a Riemannian metric on M^2.
For all $u, v \in \mathfrak{X}(M^2)$, the structure (fundamental) equations of x as immersion in $\mathbb{R}^4$, namely Gauß equation and Weingarten equation are given by:

$$\begin{aligned} \bar{\nabla}_u dx(v) &= dx(\nabla^1_u v) + \mathrm{II}(u,v)y - \mathrm{I}(u,v)x; && (2.2)\\ dy(v) &= -dx(Su). && (2.3) \end{aligned}$$

The structure equations above imply the following integrability conditions:

$$\begin{aligned} \left(\nabla^1_u S\right) v &= \left(\nabla^1_v S\right) u; \qquad \text{Codazzi equation} && (2.4)\\ R^1(u,v)w &= \mathrm{I}(w,v)u - \mathrm{I}(u,w)v + \mathrm{II}(w,v)Su - \mathrm{II}(u,w)Sv, && (2.5) \end{aligned}$$

where ∇^1 and R^1 denote the Levi-Civita connection and the Riemannian curvature tensor on M^2 of the first fundamental form, respectively. As a consequence of the equation (2.5) we have the well known relation between the scalar curvature κ of the induced metric and the Gauß-Kronecker curvature K:

$$\kappa = 1 + K. \tag{2.6}$$

Consequently, from the equation (2.1) and the Codazzi equation (2.4), one can check that the Levi-Civita connection ∇^3 of the third fundamental form is given by:

$$\nabla^3_u v = S^{-1} \nabla^1_u S v. \tag{2.7}$$

Using (2.7), one can verify, for any $u, v, w \in \mathfrak{X}(M^2)$, that

$$w\mathrm{II}(u,v) = \mathrm{II}(\nabla^1_w u, v) + \mathrm{II}(u, \nabla^3_w v). \tag{2.8}$$

Because of (2.8) the triple $(\nabla^1, \mathrm{II}, \nabla^3)$ is said to be conjugate. Therefore the Levi-Civita connection ∇^2 of the second fundamental form is

$$\nabla^2 = \frac{\nabla^1 + \nabla^3}{2}. \tag{2.9}$$

The difference tensor C defined by

$$C(u,v) := \frac{1}{2}(\nabla^1_u v - \nabla^3_u v) = -\frac{1}{2} S^{-1}(\nabla^1_u S)v \tag{2.10}$$

is a symmetric $(1,2)$-tensor field, satisfying with the there connections ∇^1, ∇^2 and ∇^3

$$\nabla^1 = \nabla^2 + C, \qquad \nabla^3 = \nabla^2 - C. \tag{2.11}$$

The Tchebychev vector field T is defined by: $\mathrm{II}(u,T) := \frac{1}{2}\mathrm{tr}[v \longmapsto C(u,v)]$. And finally, the $(1,2)$-tensor field $\widetilde{C}$, defined by

$$\widetilde{C}(u,v) :\ = C(u,v) - \frac{1}{2}[\mathrm{II}(T,v)u + \mathrm{II}(T,u)v + \mathrm{II}(u,v)T] \tag{2.12}$$

is symmetric and traceless, it is called traceless part of C. The cubic form $\widehat{C}$ defined by $\widehat{C}(u,v,w) = \mathrm{II}(C(u,v),w)$ is totally symmetric and satisfies: $2\widehat{C} = -\nabla^1\mathrm{II} = \nabla^3\mathrm{II}$.

3 Tchebychev surfaces of $\mathbb{S}^3(1)$

The following proposition from [9] gives a characterization of Tchebychev surfaces.

Proposition 3.1. *Let the immersion $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be non-degenerate and denote by div^2 the divergence with respect to the second fundamental form. The following assertions are equivalent:*

(i) *the tensor field $\nabla^2\widetilde{C}$ is totally symmetric;*

(ii) *the operator L defined by $Lu := \frac{1}{2}(Su - S^{-1}u - \nabla^2_u T)$ for $u \in \mathfrak{X}(M^2)$, is proportional to the identity map;*

(iii) $\mathrm{div}^2\,\widetilde{C} = 0$.

Definition 3.2. *A regular immersion $x\colon M^2 \longrightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4$ is called a "Tchebychev surface" (or is of Tchebychev type) if and only if it satisfies one (and therefore all) of the assertions of Proposition 3.1.*

Example 3.3. The following gives some simple examples of Tchebychev surfaces; recall from (2.12) that $C = 0$ implies $\widetilde{C} = 0$. Suppose that the regular immersion $x\colon M^2 \longrightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4$ satisfies one of the following conditions:

(i) x is isoparametric (if and only if $C = 0$, see [9]);

(ii) x satisfies $\widetilde{C} = 0$ (see [10] for non-isoparametric immersions with $\widetilde{C} = 0$),

then the immersion x is of Tchebychev type.

Proposition 3.4 ([9]). *An immersion $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ of a regular surface M^2 into $\mathbb{S}^3(1)$ with constant Gauß-Kronecker curvature is of Tchebychev type if and only if x is totally umbilical or its principal curvature functions λ_1 and λ_2 are related through the Cartan formula ($\lambda_1\lambda_2 + 1 = 0$).*

Lemma 3.5 ([1]). *If K denotes the Gauß-Kronecker curvature of a regular immersion of $\mathbb{S}^{n+1}(1)$, then*

$$nT = \frac{1}{2}\mathrm{grad}^2 \ln |K|^{-1} \tag{3.13}$$

where grad^2 is the gradient with respect to the second fundamental form.

Lemma 3.6. *For all vector fields $z, u, v, w \in \mathfrak{X}(M^n)$, one has:*

$$\begin{aligned} -\frac{1}{2}(\nabla^1_z R^1)(u,v,w) &= \mathrm{II}(w,v)SC(u,z) - \mathrm{II}(w,u)SC(v,z) \\ &\quad + \widehat{C}(v,z,w)Su - \widehat{C}(u,z,w)Sv. \end{aligned} \tag{3.14}$$

Proof. Straightforward calculation. □

Proof of Theorem 1.2.

Choose a I-orthonormal local differentiable frame (e_1, e_2) of principal vector fields ($Se_1 = \lambda_1 e_1$ and $Se_2 = \lambda_2 e_2$). Using the formula (3.14), one has:

$$\begin{aligned}
-\frac{1}{2}(\nabla^1_{e_1}R^1)(e_1,e_1,e_1) &= 0 &&= -\frac{1}{2}(\nabla^1_{e_1}R^1)(e_1,e_1,e_2);\\
-\frac{1}{2}(\nabla^1_{e_1}R^1)(e_2,e_2,e_1) &= 0 &&= -\frac{1}{2}(\nabla^1_{e_1}R^1)(e_2,e_2,e_2);\\
-\frac{1}{2}(\nabla^1_{e_2}R^1)(e_1,e_1,e_1) &= 0 &&= -\frac{1}{2}(\nabla^1_{e_2}R^1)(e_1,e_1,e_2);\\
-\frac{1}{2}(\nabla^1_{e_2}R^1)(e_2,e_2,e_1) &= 0 &&= -\frac{1}{2}(\nabla^1_{e_2}R^1)(e_2,e_2,e_2);\\
-\frac{1}{2}(\nabla^1_{e_1}R^1)(e_1,e_2,e_1) &= -2\lambda_1\lambda_2 T_1 e_2 &&= \frac{1}{2}(\nabla^1_{e_1}R^1)(e_2,e_1,e_1);\\
-\frac{1}{2}(\nabla^1_{e_1}R^1)(e_1,e_2,e_2) &= 2\lambda_1\lambda_2 T_1 e_1 &&= \frac{1}{2}(\nabla^1_{e_1}R^1)(e_2,e_1,e_2);\\
-\frac{1}{2}(\nabla^1_{e_2}R^1)(e_1,e_2,e_1) &= -2\lambda_1\lambda_2 T_2 e_2 &&= \frac{1}{2}(\nabla^1_{e_2}R^1)(e_2,e_1,e_1);\\
-\frac{1}{2}(\nabla^1_{e_2}R^1)(e_1,e_2,e_2) &= 2\lambda_1\lambda_2 T_2 e_1 &&= \frac{1}{2}(\nabla^1_{e_2}R^1)(e_2,e_1,e_2).
\end{aligned}$$

Thus $\nabla^1 R^1 = 0$ if and only if $T = 0$, where $T_1 = \lambda_1^{-1}T^1$ and $T_2 = \lambda_2^{-1}T^2$; $T = T^1 e_1 + T^2 e_2$ is the Tchebychev vector field. The second equivalence is well known.

Corollary 3.7. *Non-degenerate compact locally symmetric surfaces of $\mathbb{S}^3(1)$ are of Tchebychev type.*

Proof. Note that compact surfaces of $\mathbb{S}^3(1)$ with constant Gauß-Kronecker curvature are totally umbilic or are flat (constant curvature equal to 0, this is equivalent to the fact that the Gauß-Kronecker curvature is equal to -1), see [13]. □

Remark 3.8. It is well known that regular surfaces of the Euclidean sphere $\mathbb{S}^3(1)$ with constant Gauß-Kronecker curvature are not necessarily isoparametric ([9], [13]), Theorem 1.2 proves that there exist non-isoparametric locally symmetric surfaces of $\mathbb{S}^3(1)$, while in higher dimension non-degenerate locally symmetric hypersurfaces of the Euclidean sphere $\mathbb{S}^{n+1}(1)$ are isoparametric with at most two distinct principal curvatures, see [4], [9], [13].

Remark 3.9. Let $x\colon M^n \longrightarrow \mathbb{S}^{n+1}(1)$, $n \geq 2$ (arbitrary), be a regular immersion. One knows that the Gauß map associated to x defines a regular immersion $x^*\colon M^n \longrightarrow \mathbb{S}^{n+1}(1)$. The correspondance $x \longmapsto x^*$ is called polarization, and (x, x^*) is a polar pair [1]. From [10] the mapping $x^* \circ x^{-1}$ is geodesic if and only if x, x^* are isoparametric. Because of $C^* = -C$ and $T^* = -T$ [1] the local symmetry is polarization invariant.

4 Proof of Theorem 1.1

In this section, we prove our main result. To prove Theorem 1.1, because isoparametric surfaces have constant mean curvature and constant scalar curvature, we have only to prove that the Tchebychev type for surfaces of constant mean curvature or with shape operator of constant length implies that the two principal curvature functions are constant.

4.1 Immersions of surfaces without umbilics

Assume that the regular immersion $x\colon M^2 \longrightarrow \mathbb{S}^3(1) \subset \mathbb{R}^4$ has no umbilic points. Denote by λ_1 and λ_2 the two principal curvature functions, and choose a I-orthonormal frame (e_1, e_2) such that $Se_1 = \lambda_1 e_1$ and $Se_2 = \lambda_2 e_2$. There exist two differentiable functions $\alpha, \beta \in C^\infty(M^2)$ such that:

$$\nabla^1_{e_1} e_1 = \alpha e_2; \quad \nabla^1_{e_2} e_2 = \beta e_1. \tag{4.15}$$

Because of $\nabla^1 \mathrm{I} = 0, \mathrm{I}(e_1, e_1) = 1 = \mathrm{I}(e_2, e_2)$ and $\mathrm{I}(e_1, e_2) = 0$, one also has:

$$\nabla^1_{e_1} e_2 = -\alpha e_1; \quad \nabla^1_{e_2} e_1 = -\beta e_2. \tag{4.16}$$

So the Lie bracket of e_1 and e_2 is:

$$[e_1, e_2] = -\alpha e_1 + \beta e_2. \tag{4.17}$$

With respect to the frame (e_1, e_2), the structure equations of x as immersion in $\mathbb{R}^4$ are:

$$\begin{cases} \bar{\nabla}_{e_1} dx(e_1) = \alpha dx(e_2) + \lambda_1 y - x; \\ \bar{\nabla}_{e_1} dx(e_2) = -\alpha dx(e_1); \\ \bar{\nabla}_{e_2} dx(e_1) = -\beta dx(e_2); \\ \bar{\nabla}_{e_2} dx(e_2) = \beta dx(e_1) + \lambda_2 y - x; \\ dy(e_1) = -\lambda_1 e_1; \qquad dy(e_2) = -\lambda_2 e_2. \end{cases} \tag{4.18}$$

Proposition 4.1. *The functions α, β, λ_1 and λ_2 satisfy the following first order partial differential equations:*

$$\begin{aligned} e_1(\beta)+e_2(\alpha) &= \alpha^2+\beta^2+1+\lambda_1\lambda_2; \quad (4.19)\\ e_1(\lambda_2) &= \beta(\lambda_2-\lambda_1); \quad (4.20)\\ e_2(\lambda_1) &= \alpha(\lambda_1-\lambda_2). \quad (4.21)\end{aligned}$$

Proof. From the integrability condition (2.5), one has:

$$R^1(e_1,e_2)e_2 = e_1+\lambda_1\lambda_2 e_1 = (1+\lambda_1\lambda_2)e_1. \quad (4.22)$$

The equation (4.19) is then a consequence of the equations (4.15), (4.16), (4.17) and (4.22). The equations (4.20) and (4.20) are consequences of the Codazzi equation (2.4). □

Proposition 4.2. *With repect to the frame (e_1, e_2), the components of the tensor field C and the Tchebychev vector field T are given by:*

$$\begin{aligned} C^1_{11} &= -\frac{1}{2}\lambda_1^{-1}e_1(\lambda_1); & C^2_{11} &= -\frac{1}{2}\lambda_2^{-1}\alpha(\lambda_1-\lambda_2);\\ C^1_{22} &= -\frac{1}{2}\lambda_1^{-1}\beta(\lambda_2-\lambda_1); & C^2_{22} &= -\frac{1}{2}\lambda_2^{-1}e_2(\lambda_2); \quad (4.23)\\ C^1_{12} &= -\frac{1}{2}\lambda_1^{-1}\alpha(\lambda_1-\lambda_2); & C^2_{12} &= -\frac{1}{2}\lambda_2^{-1}\beta(\lambda_2-\lambda_1);\end{aligned}$$

$$T = -\frac{1}{4}[(\lambda_1^{-2}e_1(\lambda_1)+\lambda_1^{-1}\lambda_2^{-1}\beta(\lambda_2-\lambda_1))e_1+(\lambda_1^{-1}\lambda_2^{-1}\alpha(\lambda_1-\lambda_2)+\lambda_2^{-2}e_2(\lambda_2))e_2]. \quad (4.24)$$

4.2 Tchebychev surfaces of constant mean curvature

Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be a regular immersion without umbilics. Assume that the immersion x has constant mean curvature H. Denote by $\lambda_1=\lambda$ the first principal curvature ($Se_1=\lambda e_1$ and $Se_2=(2H-\lambda)e_2$).

Proposition 4.3.

(i) *The following holds (Integrability conditions for surfaces immersed in $\mathbb{S}^3(1)$ with constant mean curvature):*

$$\begin{aligned} e_1(\beta)+e_2(\alpha) &= \alpha^2+\beta^2+1+\lambda(2H-\lambda); \quad (4.25)\\ e_1(\lambda) &= 2\beta(\lambda-H); \quad (4.26)\\ e_2(\lambda) &= 2\alpha(\lambda-H). \quad (4.27)\end{aligned}$$

(ii) *The Tchebychev vector field T for the immersion x is given by:*

$$T=(\lambda-H)^2\lambda^{-2}(2H-\lambda)^{-2}((2H-\lambda)\beta e_1+\lambda\alpha e_2) \quad (4.28)$$

(iii) *The Levi-Civita connection of the second fundamental form is given by:*

$$\begin{aligned} \nabla^2_{e_1}e_1 &= \beta\lambda^{-1}(\lambda-H)e_1+\alpha H(2H-\lambda)^{-1}e_2;\\ \nabla^2_{e_1}e_2 &= -\alpha H\lambda^{-1}e_1+\beta(2H-\lambda)^{-1}(H-\lambda)e_2; \quad (4.29)\\ \nabla^2_{e_2}e_1 &= \alpha\lambda^{-1}(\lambda-H)e_1-\beta(2H-\lambda)^{-1}He_2\\ \nabla^2_{e_2}e_2 &= \beta\lambda^{-1}He_1-\alpha(2H-\lambda)^{-1}(\lambda-H)e_2.\end{aligned}$$

Proposition 4.4. *The regular immersion x of constant mean curvature H (without umbilics) is of Tchebychev type if and only if the following first order partial differential equations are satisfied:*

$$e_1(\alpha) = -\alpha\beta\lambda^{-1}(2H-\lambda)^{-1}(\lambda^2 - 2H\lambda + 6H^2); \tag{4.30}$$

$$e_2(\beta) = -\alpha\beta\lambda^{-1}(2H-\lambda)^{-1}(\lambda^2 - 2H\lambda + 6H^2); \tag{4.31}$$

$$2He_1(\beta) = \frac{\lambda(3H-\lambda)(1+2H\lambda-\lambda^2)}{(\lambda-H)} + \frac{6\alpha^2H^2}{(2H-\lambda)} + \frac{2\beta^2H(-3H+\lambda)}{\lambda}; \tag{4.32}$$

$$2He_2(\alpha) = -\frac{(2H-\lambda)(H+\lambda)(1+2H\lambda-\lambda^2)}{(\lambda-H)} + \frac{6\beta^2H^2}{\lambda} - \frac{2\alpha^2H(H+\lambda)}{(2H-\lambda)}. \tag{4.33}$$

Proof. Using (4.28) and (4.29), one has:

$$\begin{aligned}
\nabla_1^2T^1 &= \mathrm{I}(\nabla^2_{e_1}T, e_1)\\
&= \beta^2(\lambda-H)^2(2H-\lambda)^{-2}\lambda^{-3}(\lambda^2-3H\lambda+6H^2)\\
&\quad -\alpha^2H(\lambda-H)^2(2H-\lambda)^{-2}\lambda^{-2} + (\lambda-H)^2(2H-\lambda)^{-1}\lambda^{-2}e_1(\beta);\\
\nabla_1^2T^2 &= \mathrm{I}(\nabla^2_{e_1}T, e_2)\\
&= \alpha\beta(\lambda-H)^2(2H-\lambda)^{-3}\lambda^{-2}(\lambda^2-2H\lambda+6H^2)\\
&\quad +(\lambda-H)^2(2H-\lambda)^{-2}\lambda^{-1}e_1(\alpha);\\
\nabla_2^2T^1 &= \mathrm{I}(\nabla^2_{e_2}T, e_1)\\
&= \alpha\beta(\lambda-H)^2(2H-\lambda)^{-2}\lambda^{-3}(\lambda^2-2H\lambda+6H^2)\\
&\quad +(\lambda-H)^2(2H-\lambda)^{-1}\lambda^{-2}e_2(\beta);\\
\nabla_2^2T^2 &= \mathrm{I}(\nabla^2_{e_2}T, e_2)\\
&= \alpha^2(\lambda-H)^2(2H-\lambda)^{-3}\lambda^{-2}(\lambda^2-H\lambda+4H^2)\\
&\quad -\beta^2H(\lambda-H)^2(2H-\lambda)^{-2}\lambda^{-2} + (\lambda-H)^2(2H-\lambda)^{-2}\lambda^{-1}e_2(\alpha).
\end{aligned}$$

The immersion is of Tchebychev type iff

$$\mathrm{I}(Le_1, e_2) = 0 = \mathrm{I}(Le_2, e_1) \quad \text{and} \quad \mathrm{I}(Le_1, e_1) = \mathrm{I}(Le_2, e_2)$$

iff

$$\nabla_1^2T^2 = 0 = \nabla_2^2T^1 \quad \text{and} \quad \lambda - \lambda^{-1} - \nabla_1^2T^1 = 2H - \lambda - (2H-\lambda)^{-1} - \nabla_2^2T^2.$$

One has: $\mathrm{I}(Le_1, e_2) = 0 = \mathrm{I}(Le_2, e_1)$ iff the equations (4.30) and (4.31) are valid. And $\mathrm{I}(Le_1, e_1) = \mathrm{I}(Le_2, e_2)$ iff the derivatives $e_1(\beta)$ and $e_2(\alpha)$ satisfy the following equation:

$$\begin{aligned}
\lambda e_2(\alpha) - (2H-\lambda)e_1(\beta) &= \left(\frac{\beta^2}{\lambda} - \frac{\alpha^2}{2H-\lambda}\right)(\lambda^2-2H\lambda+6H^2)\\
&\quad -2(1+2H\lambda-\lambda^2)\lambda(2H-\lambda)(\lambda-H)^{-1}.
\end{aligned} \tag{4.34}$$

The equation (4.34) together with the integrability condition (4.25) implies that $\mathrm{I}(Le_1, e_1) = \mathrm{I}(Le_2, e_2)$ if and only if the equations (4.32) and (4.33) are valid. □

Corollary 4.5. *Minimal Tchebychev surfaces of $\mathbb{S}^3(1)$ are parts of the Clifford torus.*

Proof. If $H = 0$, (4.32) and (4.33) imply: $1 - \lambda^2 = 0$ everywhere. $\square$

Suppose now that the Tchebychev immersion x (without umbilics) has constant mean curvature $H \neq 0$. One has the following derivatives:

$$\begin{aligned} e_1(\beta) &= \frac{\lambda(3H-\lambda)(1+2H\lambda-\lambda^2)}{2H(\lambda-H)} + \frac{3\alpha^2 H}{2H-\lambda} + \frac{\beta^2(-3H+\lambda)}{\lambda}; && (4.35)\\ e_2(\alpha) &= \frac{-(2H-\lambda)(\lambda+H)(1+2H\lambda-\lambda^2)}{2H(\lambda-H)} - \frac{\alpha^2(H+\lambda)}{(2H-\lambda)} + \frac{3\beta^2 H}{\lambda}. && (4.36)\end{aligned}$$

Lemma 4.6. *Assume that an immersion of a Tchebychev surface has constant mean curvature $H \neq 0$. If one of the functions α and β vanishes on an open subset U of M^2, both α and β vanish on U.*

Proof. Assume that $\alpha = 0$ on an open subset U of M^2. From (4.36), one has:

$$\beta^2 = \frac{\lambda(2H-\lambda)(\lambda+H)(1+2H\lambda-\lambda^2)}{6H^2(\lambda-H)}, \tag{4.37}$$

which is a function of λ. Derivating (4.37) with respect to e_1 and substituting (4.37) into the equation $e_1(\beta) = \beta^2 + 1 + 2H\lambda - \lambda^2$, one gets the following equation:

$$3\lambda^5 - 11H\lambda^4 + (18H^2-1)\lambda^3 - 3H(6H-1)\lambda^2 + 2H^2(2H^2-5)\lambda + 4H^2 = 0. \tag{4.38}$$

The equation (4.38) shows that λ everywhere solves the same fifth order polynomial equation with constant coefficients, which implies that λ is constant on U. Therefore β is equal to 0; this proves the lemma. $\square$

Theorem 4.7. *An immersion $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ of a regular surface with constant mean curvature $H \neq 0$, without umbilics in $\mathbb{S}^3(1)$, is of Tchebychev type if and only if this immersion is isoparametric and of Gauß-Kronecker curvature equal to -1.*

Proof. We want to prove that the functions α and β vanish on M^2. Suppose that it is not true that α and β vanish on M^2. There exists a point $p \in M^2$ such that at least one of α and β does not vanish at p. By continuity and using Lemma 4.6, one can conclude that there exists an open subset U of M^2 in which $\alpha \neq 0$ and $\beta \neq 0$. On U, using the four derivatives $e_1(\alpha)$, $e_2(\beta)$, $e_1(\beta)$ and $e_2(\alpha)$, one has:

$$\begin{aligned} 0 &= e_1e_2(\beta) - e_2e_3(\beta) - [e_1,e_2](\beta)\\ &= 6\alpha H(2H-\lambda)^{-2}\lambda^{-1}(3H-\lambda)(\lambda-H)(\alpha^2+\beta^2) + 2\alpha\lambda H^{-1}(\lambda-H)(3H-\lambda)\\ &\quad +\alpha(\lambda-H)^{-1}(2H-\lambda)^{-1}(1+2H\lambda-\lambda^2)(-4\lambda^2+5H\lambda+3H^2)\end{aligned}$$

and

$$\begin{aligned} 0 &= e_1e_2(\alpha) - e_2e_3(\alpha) - [e_1,e_2](\alpha)\\ &= 6\beta H(2H-\lambda)^{-1}\lambda^{-2}(H+\lambda)(\lambda-H)(\alpha^2+\beta^2) + 2\beta H^{-1}(\lambda-H)(\lambda+\lambda)(2H-\lambda)\\ &\quad +\beta(\lambda-H)^{-1}\lambda^{-1}(1+2H\lambda-\lambda^2)(-4\lambda^2+11H\lambda-3H^2).\end{aligned}$$

So the functions α, β and λ are related through the following:

$$3H^2\lambda^{-2}(2H-\lambda)^{-2}(\lambda-H)^2(\alpha^2+\beta^2)+(\lambda-H)^2=\lambda^2-2H\lambda-1. \tag{4.39}$$

This simplifies to

$$3H^2\lambda^{-2}(2H-\lambda)^{-2}(\lambda-H)^2(\alpha^2+\beta^2)+H^2+1=0, \tag{4.40}$$

which is absurd. Therefore α and β vanish identically. This proves that λ is constant. □

Corollary 4.8. *Assume that a Tchebychev surface M^2 of $\mathbb{S}^3(1)$ has constant mean curvature. If M^2 contains an umbilic point, then $x(M^2)$ is contained in a geodesic sphere.*

4.3 Tchebychev surfaces whose shape operators have constant length

Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be a regular immersion without umbilics. Denote by λ_1 and λ_2 the two principal curvature functions. Assume that the immersion x has non-vanishing mean curvature function ($H \neq 0$ everywhere) and the shape operator S has constant length $\|S\|^2=\lambda_1^2+\lambda_2^2=a^2$, $0<a\in\mathbb{R}$. Under these assumptions there is a differentiable function $\gamma\colon M^2 \longrightarrow]0,\frac{\pi}{4}[\cup]\frac{\pi}{4},\frac{\pi}{2}[\cup]\frac{\pi}{2},\frac{3\pi}{4}[\cup]\frac{3\pi}{4},\pi[$ such that:

$$\lambda_1=a\cos\gamma \quad \text{and} \quad \lambda_2=a\sin\gamma. \tag{4.41}$$

Proposition 4.9. *With respect to the frame (e_1,e_2) from Section 4.1, the following holds:*

(i) *Integrability conditions (for immersions with shape operator of constant length):*

$$\begin{aligned} e_1(\beta)+e_2(\alpha) &= \alpha^2+\beta^2+1+\frac{1}{2}a^2\sin 2\gamma; && (4.42)\\ e_1(\gamma) &= \beta(\tan\gamma-1); && (4.43)\\ e_2(\gamma) &= \alpha(1-\cot\gamma). && (4.44)\end{aligned}$$

(ii) *Components of C:*

$$\begin{aligned} C_{11}^1 &= \frac{1}{2}\beta\tan\gamma(\tan\gamma-1); & C_{11}^2 &= \frac{1}{2}\alpha(1-\cot\gamma);\\ C_{22}^2 &= \frac{1}{2}\alpha\cot\gamma(\cot\gamma-1); & C_{22}^1 &= \frac{1}{2}\beta(1-\tan\gamma);\\ C_{12}^1 &= \frac{1}{2}\alpha(\tan\gamma-1); & C_{12}^2 &= \frac{1}{2}\beta(\cot\gamma-1).\end{aligned}$$

(iii) *Tchebychev vector field $T=T^1e_1+T^2e_2$:*

$$T=\frac{\beta\cos 2\gamma(\cot\gamma-1)}{4a\cos^3\gamma}e_1+\frac{\alpha\cos 2\gamma(1-\tan\gamma)}{4a\sin^3\gamma}e_2 \tag{4.45}$$

(iv) *Connection ∇^2 of the second fundamental form:*

$$\begin{aligned} \nabla^2_{e_1}e_1 &= -\frac{1}{2}\beta\tan\gamma(\tan\gamma-1)e_1+\frac{1}{2}\alpha(1+\cot\gamma)e_2; && (4.46)\\ \nabla^2_{e_1}e_2 &= -\frac{1}{2}\alpha(1+\tan\gamma)e_1-\frac{1}{2}\beta(\cot\gamma-1)e_2; && (4.47)\\ \nabla^2_{e_2}e_1 &= -\frac{1}{2}\alpha(\tan\gamma-1)e_1-\frac{1}{2}\beta(\cot\gamma+1)e_2; && (4.48)\\ \nabla^2_{e_2}e_2 &= \frac{1}{2}\beta(1+\tan\gamma)e_1-\frac{1}{2}\alpha\cot\gamma(\cot\gamma-1)e_2. && (4.49)\end{aligned}$$

Corollary 4.10. *Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be a regular immersion of M^2 into $\mathbb{S}^3(1)$. Assume that x admits no umbilics and the mean curvature function vanishes nowhere on M^2. Then the immersion x is of Tchebychev type if and only if the following equations in terms of the frame (e_1, e_2) from section 4.1 are valid:*

$$
\begin{aligned}
e_1(\alpha) &= -\frac{\alpha\beta(15+4\cos 2\gamma+\cos 4\gamma+6\sin 2\gamma-\sin 4\gamma)}{8\sin\gamma\cos^2\gamma(\cos\gamma+\sin\gamma)};\\
e_2(\beta) &= -\frac{\alpha\beta(15-4\cos 2\gamma+\cos 4\gamma+6\sin 2\gamma+\sin 4\gamma)}{8\cos\gamma\sin^2\gamma(\cos\gamma+\sin\gamma)};\\
e_1(\beta) &= \frac{(a^2\sin\gamma\cos\gamma+1)(2\sin^2\gamma+1)\cos^2\gamma}{\cos 2\gamma}+\frac{\beta^2(2\cos\gamma+5\cos 3\gamma-16\sin\gamma+4\sin 3\gamma)}{8\cos^2\gamma(\cos\gamma+\sin\gamma)}\\
&\quad+\frac{\alpha^2(18\cos\gamma+2\cos 3\gamma-4\cos^2\gamma(-5\sin\gamma+\sin 3\gamma))}{8\sin^2\gamma(\cos\gamma+\sin\gamma)};\\
e_2(\alpha) &= \frac{-(a^2\sin\gamma\cos\gamma+1)(2\cos^2\gamma+1)\sin^2\gamma}{\cos 2\gamma}-\frac{\alpha^2(18\cos\gamma+\cos 3\gamma+\cos 5\gamma+8\sin\gamma)}{8\sin^2\gamma(\cos\gamma+\sin\gamma)}\\
&\quad-\frac{\beta^2(-4\cos\gamma+3\cos 3\gamma+\cos 5\gamma-18\sin\gamma+2\sin 3\gamma)}{8\cos^2\gamma(\cos\gamma+\sin\gamma)}.
\end{aligned}
$$

Proof. Using (4.45), (4.46), (4.47), (4.48) and (4.49), one has

$$\nabla^2_{e_1}T = (\nabla^2_1 T^1)e_1 + (\nabla^2_{e_1}T^2)e_2 \text{ and } \nabla^2_{e_2}T = (\nabla^2_2 T^1)e_1 + (\nabla^2_{e_2}T^2)e_2,$$

where:

$$
\begin{aligned}
4a\nabla^2_1 T^1 &= e_1 T^1 - \frac{1}{2}T^1\beta\tan\gamma(\tan\gamma-1) - \frac{1}{2}\alpha T^2(1+\tan\gamma)\\
&= \beta^2(\tan\gamma-1)\left\{\frac{-2\sin 2\gamma(\cot\gamma-1)}{\cos^3\gamma}-\frac{\cos 2\gamma}{\sin^2\gamma\cos^3\gamma}+\frac{3\cos 2\gamma\sin\gamma(\cot\gamma-1)}{\cos^4\gamma}\right\}\\
&\quad-\frac{\beta^2\cos 2\gamma(\cot\gamma-1)(\tan\gamma-1)\tan\gamma}{2\cos^3\gamma}-\frac{\alpha^2\cos 2\gamma(1-\tan^2\gamma)}{2\sin^3\gamma}\\
&\quad+\frac{\cos 2\gamma(\cot\gamma-1)}{\cos^3\gamma}e_1(\beta);
\end{aligned}
$$

$$
\begin{aligned}
4a\nabla^2_1 T^2 &= e_1 T^2 + \frac{1}{2}T^1\alpha(1+\cot\gamma) - \frac{1}{2}\beta T^2(\cot\gamma-1)\\
&= \alpha\beta(\tan\gamma-1)\left\{\frac{-2\sin 2\gamma(1-\tan\gamma)}{\sin^3\gamma}-\frac{cos2\gamma}{\sin^3\gamma\cos^2\gamma}-\frac{3\cos 2\gamma(1-\tan\gamma)\cos\gamma}{\sin^4\gamma}\right.\\
&\quad\left.+\frac{\cos 2\gamma(\cot^2\gamma-1)}{2(\tan\gamma-1)\cos^3\gamma}+\frac{\cos 2\gamma(\cot\gamma-1)}{2\sin^3\gamma}\right\}+\frac{\cos 2\gamma(1-\tan\gamma)}{\sin^3\gamma}e_1(\alpha);
\end{aligned}
$$

$$
\begin{aligned}
4a\nabla^2_2 T^1 &= e_2 T^1 - \frac{1}{2}T^1\alpha(\tan\gamma-1) + \frac{1}{2}\beta T^2(1+\tan\gamma)\\
&= \alpha\beta(1-\cot\gamma)\left\{\frac{-2\sin 2\gamma(\cot\gamma-1)}{\cos^3\gamma}-\frac{cos2\gamma}{\sin^2\gamma\cos^3\gamma}+\frac{3\cos 2\gamma(\cot\gamma-1)\sin\gamma}{\cos^4\gamma}\right.\\
&\quad\left.+\frac{\cos 2\gamma(1-\tan^2\gamma)}{2(1-\cot\gamma)\sin^3\gamma}-\frac{\cos 2\gamma(\tan\gamma-1)}{2\cos^3\gamma}\right\}+\frac{\cos 2\gamma(\cot\gamma-1)}{\cos^3\gamma}e_2(\beta);
\end{aligned}
$$

$$
\begin{aligned}
4a\nabla_2^2 T^2 &= e_2 T^2 - \frac{1}{2}T^2\alpha\cot\gamma(\cot\gamma - 1) - \frac{1}{2}\beta T^1(\cot\gamma + 1) \\
&= \alpha^2(1-\cot\gamma)\left\{\frac{-2\sin 2\gamma(1-\tan\gamma)}{\sin^3\gamma} - \frac{\cos 2\gamma}{\sin^3\gamma\cos^2\gamma} - \frac{3\cos 2\gamma\cos\gamma(1-\tan\gamma)}{\sin^4\gamma}\right\} \\
&\quad -\frac{\beta^2\cos 2\gamma(\cot^2\gamma - 1)}{2\cos^3\gamma} - \frac{\alpha^2\cos 2\gamma(1-\tan\gamma)(\cot\gamma - 1)\cot\gamma}{2\sin^3\gamma} \\
&\quad +\frac{\cos 2\gamma(1-\tan\gamma)}{\sin^3\gamma}e_2(\alpha).
\end{aligned}
$$

The immersion is of Tchebychev type if and only if the following equations are fulfilled:

$$
\mathrm{I}(Le_1, e_2) = 0 = \mathrm{I}(e_1, Le_2); \qquad \mathrm{I}(Le_1, e_1) = \mathrm{I}(Le_2, e_2);
$$

These equations are equivalent to the following:

$$
\nabla_1^2 T^2 = 0 = \nabla_2^2 T^1; \tag{4.50}
$$

$$
a\cos\gamma - \frac{1}{a\cos\gamma} - \nabla_1^2 T^1 = a\sin\gamma - \frac{1}{a\sin\gamma} - \nabla_2^2 T^2 \tag{4.51}
$$

From (4.50), one has the derivatives $e_1(\alpha)$ and $e_2(\beta)$. And the equation (4.51) provides the following equation in $e_1(\beta)$ and $e_2(\alpha)$:

$$
\begin{aligned}
&\frac{4(\cos\gamma - \sin\gamma)(a^2\cos\gamma\sin\gamma + 1)}{\cos\gamma\sin\gamma} = 4a(\nabla_1^2 T^1 - \nabla_2^2 T^2) \\
= &\frac{\cos 2\gamma(\cot\gamma - 1)}{\cos^3\gamma}e_1(\beta) - \frac{\cos 2\gamma(1-\tan\gamma)}{\sin^3\gamma}e_2(\alpha) \\
&+\beta^2(\tan\gamma - 1)\left(\frac{-2\sin 2\gamma(\cot\gamma - 1)}{\cos^3\gamma} - \frac{\cos 2\gamma}{\sin^2\gamma\cos^3\gamma} + \frac{5\cos 2\gamma\sin\gamma(\cot\gamma - 1)}{2\cos^4\gamma}\right. \\
&\left.+\frac{\cos 2\gamma(\cot^2\gamma - 1)}{2\cos^3\gamma(\tan\gamma - 1)}\right) + \alpha^2(1-\cot\gamma)\left(\frac{2\sin 2\gamma(1-\tan\gamma)}{\sin^3\gamma} + \frac{\cos 2\gamma}{\sin^3\gamma\cos^2\gamma}\right. \\
&\left.+\frac{5\cos 2\gamma\cos\gamma(1-\tan\gamma)}{2\sin^4\gamma} - \frac{\cos 2\gamma(1-\tan^2\gamma)}{2\sin^3\gamma(1-\cot\gamma)}\right)
\end{aligned}
$$

From the equations (4.42) and (4.51), one gets the derivatives $e_1(\beta)$ and $e_2(\alpha)$. □

Lemma 4.11. *Suppose that the regular immersion x has no umbilics and satisfies $\|S\|^2 = a^2 = cst$. If one of the functions α and β vanishes on an open subset U, then both α and β like γ are constant on U.*

Proof. Fix $\alpha = 0$ on an open subset U, and suppose that β does not vanish identically on U. By continuity, U contains an open subset V on which $\beta \neq 0$. From the equations of Corollary 4.10 with $\alpha = 0$, β^2 and γ are related on V as follows:

$$
\beta^2 = \frac{2(2+\cos 2\gamma)(2+a^2\sin 2\gamma)\sin\gamma\cos^2\gamma}{(\cos\gamma - \sin\gamma)(8 - 2\cos 2\gamma + 4\sin 2\gamma + \sin 4\gamma)}. \tag{4.52}
$$

Deriving the equation (4.52) with respect to e_1 and using the integrability condition (4.42), one has:

$$
\begin{aligned}
&20(44+15a^2)\cos\gamma + (416 - 76a^2)\cos 3\gamma - (210a^2 + 32)\cos 5\gamma + (2a^2 - 104)\cos 7\gamma \\
&\quad -(8+14a^2)\cos 9\gamma - 2a^2\cos 11\gamma + (126a^2 + 104)\sin\gamma + (278a^2 + 992)\sin 3\gamma \\
&\quad +(181a^2 - 64)\sin 5\gamma + (12+9a^2)\sin 7\gamma + (4 - 21a^2)\sin 9\gamma - a^2\sin 11\gamma = 0.
\end{aligned}
$$

Since for any integer r, $\cos r\gamma$ and $\sin r\gamma$ are polynomial of order r in $\cos\gamma$ and $\sin\gamma$, the equation above shows that $\cos\gamma$ and $\sin\gamma$ everywhere are solutions of the same $11th$ order polynomial equation with constant coefficients. So $\cos\gamma$ and $\sin\gamma$ are constant, and so are γ and β, which is a contradiction to $\beta \neq 0$. □

Theorem 4.12. *Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be an immersion of a regular surface M^2 into $\mathbb{S}^3(1)$ without umbilics and such that the mean curvature function vanishes nowhere and the shape operator has constant squared length, then the immersion x is of Tchebychev type if and only if x is isoparametric.*

Proof. We want to prove that under these assumptions, α and β must vanish on M^2. Suppose that α and β simultaneously are not zero somewhere. From Lemma 4.11, there exists an open subset U of M^2 on which both α and β are not zero. On U, both the following equations

$$0 = \frac{1}{\beta}\left(e_1e_2(\alpha) - e_2e_1(\alpha) - [e_1,e_2](\alpha)\right); \tag{4.53}$$

$$0 = \frac{1}{\alpha}\left(e_1e_2(\beta) - e_2e_1(\beta) - [e_1,e_2](\beta)\right), \tag{4.54}$$

are linear in α^2 and β^2. Solving these equations α^2 and β^2 are then functions of γ: $\alpha^2 = f(\gamma)$ and $\beta^2 = g(\gamma)$. Commputing now the derivative of β^2 with respect to e_2, in two different ways, yields

$$g'(\gamma)e_2(\gamma) = 2\beta e_2(\beta). \tag{4.55}$$

Rewriting this in the form

$$g'(\gamma)\frac{e_2(\gamma)}{\alpha} = 2\beta^2\frac{e_2(\beta)}{\alpha\beta}, \tag{4.56}$$

it follows that equation (4.56) can be expressed in terms of γ. Looking at the explicit expression for γ from (4.56), it follows that γ has to be constant. □

Note: The calculations in the proof of Theorem 4.12 were done using Mathematica.

Corollary 4.13. *Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be an immersion of a Tchebychev surface M^2 into $\mathbb{S}^3(1)$ such that the shape operator has constant squared length. If moreover M^2 contains an umbilic point, then $x(M^2)$ contained in a geodesic sphere.*

Corollary 4.14. *Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be an immersion of a Tchebychev surface M^2 into $\mathbb{S}^3(1)$ such that the shape operator has constant squared length. If moreover the mean curvature function vanishes somewhere on M^2, then $x(M^2)$ is part of a Clifford torus.*

5 Survey of types of Tchebychev surfaces of $\mathbb{S}^3(1)$

Theorem 5.1 ([10]). *Let $x\colon M^2 \longrightarrow \mathbb{S}^3(1)$ be a regular surface immersion of a connected and orientable 2-dimensional C^∞-manifold M^2 into $\mathbb{S}^3(1)$ without umbilics. The immersion x satifies the equation $\widetilde{C} = 0$ if and only if either x is isoparametric or there exist an open interval I of constant sign and constants $k_1, k_2, k_3 \in \mathbb{R}$ such that:*

(i) $k_2 \neq 0 \neq k_1$, $k_3 < 1$; $1 - 2k_3 - k_1 > 0$, $k_3^2 + k_1 > 0$;

(ii) $1 - k_1u^2 > 0$, $k_1u^4 - 2k_3u^2 - 1 > 0$, *for all* $u \in I$;

(iii) *$x(M^2)$ is a part of the following surface in $\mathbb{S}^3(1)$:*

$$\mathbb{L} = \{\rho^{-1} k_4^{-1} (C_1 \cos k_4 v + C_2 \sin k_4 v) + C_3 \cos \frac{1}{2}\gamma(u) + C_4 \sin \frac{1}{2}\gamma(u) : (u,v) \in I \times \mathbb{R}\},$$

where: $C_1 = (1,0,0,0)$, $C_2 = (0,1,0,0)$, $C_3 = (0,0,\sqrt{\frac{1-k_3}{1-2k_3-k_1}},0)$,
$C_4 = (0,0,\sqrt{\frac{k_3^2+k_1}{(1-2k_3-k_1)(1-k_3)}},\sqrt{\frac{1}{1-k_3}})$; $k_4^{-2} = k_2^2(1-2k_3-k_1)$;
$k_4^{-2}\rho^{-2}(u) = \frac{k_1+k_3}{2k_3+k_1-1} + \frac{\sqrt{k_1+k_3^2}}{1-2k_3-k_1}\sin\gamma(u)$ *and* $\gamma(u) = \arctan(\frac{1+k_3u^2}{\sqrt{k_1u^4-2k_3u^2-1}})$.

This allows to fill the following list for Tchebychev surfaces of $\mathbb{S}^3(1)$ satisfying at least one of the conditions on the first column.

Tchebychev surfaces of $\mathbb{S}^3(1)$

<table>
<tr><td>3.3: λ_1, λ_2 const.</td><td>$C=0$
$T=0$
$\widetilde{C}=0$</td><td>isop. imm.</td><td>locally symm. imm.</td></tr>
<tr><td rowspan="2">1.2: $\lambda_1\lambda_2$ const.</td><td rowspan="2">$T=0$</td><td>isop. imm.</td><td rowspan="2">locally symm. imm.</td></tr>
<tr><td>imm. with $\lambda_1\lambda_2+1=0$
(non-isop. examples [9], [13])</td></tr>
<tr><td>4.2: $\lambda_1+\lambda_2$ const.</td><td>$C=0$
$T=0$
$\widetilde{C}=0$</td><td>isop. imm.</td><td rowspan="2">locally symm. imm.</td></tr>
<tr><td>4.3: $\lambda_1^2+\lambda_2^2$ const.</td><td>$C=0$
$T=0$
$\widetilde{C}=0$</td><td>isop. imm.</td></tr>
<tr><td rowspan="2">5.1: $\widetilde{C}=0$</td><td>$T=0$</td><td>isop. imm.</td><td>locally symm. imm.</td></tr>
<tr><td>$T\neq 0$</td><td>non-isop. examples and
classification in [10]</td><td>non-locally symm. imm.</td></tr>
</table>

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